The Motion of Viscous Liquid due to Uniform and Periodic Motion Maintained over a Segment of an Infinite Plane Boundary.

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The author was led to a consideration of the problems whose solutions are presented in this paper by the attempt to solve a certain problem in the theory of lubrication. But it was thought that these solutions have sufficient intrinsic interest to warrant their separate publication as examples in theoretical hydrodynamics, the more physical applications to lubrication being reserved for further development.

In the first three problems infinite viscous liquid is maintained in motion by uniform or periodic tangential motion of a part (a strip of infinite length) of its plane boundary. In the remaining two problems the liquid is further limited by a second plane boundary parallel to the first.

A departure, which may be more or less important, from the usual actual physical conditions is necessitated by the result that it is impossible to maintain the condition of no relative velocity at the boundary. The treatment of the problem with the condition of no slip leads to a discontinuity in the velocity of the liquid at the boundaries of the moving strip. But, on the other hand, from the nature of the problem this condition of no slip is an impossible physical state of affairs in the neighbourhood of two lines of the boundary. This complexity is avoided if we take the more general condition of slip with tangential traction proportional to the relative velocity, and then the case of no slip will be approached by taking the ratio of traction to slip to increase indefinitely. There are two other alternative suppositions which might be made and which seem to carry with them a more physical appearance. These will have to be examined before further applications are made of these solutions, but the solutions in their present form are given for their own interest.

Problem 1.

In this and the following calculations the motion is two-dimensional, and will therefore be considered to take place in the x, y plane; accordingly, the third co-ordinate will never be mentioned.

Viscous liquid occupies the part of the x, y plane for which y is positive, being bounded by the axis of x. This boundary is fixed, except that between $x = \pm\xi$ it is given a uniform velocity ϖ in its own direction. The effect may be produced by supposing the part of the axis $-\xi < x < \xi$ cut away, and underneath in contact with it a second boundary is placed which moves with the given velocity. It is required to determine the motion which is thus maintained.

The motion is steady and is, moreover, assumed to be slow.* It follows that the stream function satisfies the equation $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \psi = 0$.

If we assume a disturbance depending on $\cos kx$, then

$$\psi = (\mathrm{A}e^{-ky}+\mathrm{B}ye^{-ky})\cos kx,$$

where k is positive. This form of the stream function is appropriate to a velocity along the bounding plane varying as $\cos kx$. Let the velocity of the boundary in its own direction be $\varpi_k \cos kx$.

It is impossible to satisfy the condition of no relative motion at the boundary owing to the discontinuity that would arise at the points $x = \pm\xi$, $y = 0$ in the final synthesis of the harmonic elements. Accordingly, the assumption is made that the boundary exerts a traction on the liquid proportional to the relative velocity. Thus

$$\frac{\partial\psi}{\partial y}-\lambda\frac{\partial^2\psi}{\partial y^2} = \varpi_k \cos kx \quad (y = 0),$$

is the appropriate condition. If λ be made very small the condition of no relative velocity is approximately satisfied in general.

The remaining boundary condition is $\psi = 0 \quad (y = 0)$.

These lead to

$$\mathrm{A} = 0, \qquad \mathrm{B} = \varpi_k/(1+2\lambda k).$$

If the distribution of velocity of the boundary in its own direction be given by $f(x)$, then

$$\psi = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{\infty} \frac{f(\alpha)}{1+2\lambda k}\, ye^{-ky} \cos k(x-\alpha)\,d\alpha.$$

Suppose $\quad f(\alpha) = \varpi, \qquad -\xi < x < \xi,$

$\qquad f(\alpha) = 0, \qquad x < -\xi, \qquad x > \xi.$

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* The fact that the motion has to be assumed slow in order to obtain linear equations is a very serious limitation on the interest of the solution; but it must be noticed that when it is a question of motion between parallel planes which are very near together the non-linear terms are neglected by Osborne Reynolds in his paper on lubrication, for physical reasons, and the limitation of slow motion is thus removed. This consideration applies to the last two problems.

Hence

$$\psi = y\frac{\varpi}{\pi}\int_0^\infty \frac{e^{-ky}}{k(1+2\lambda k)}[\sin k(\xi+x)+\sin k(\xi-x)]\,dk$$

$$= y\frac{\varpi}{\pi}\left\{\tan^{-1}\frac{\xi+x}{y}+\tan^{-1}\frac{\xi-x}{y}\right.$$

$$\left.-\int_0^\infty \frac{2\lambda e^{-ky}}{1+2\lambda k}[\sin k(\xi+x)+\sin k(\xi-x)]\,dk\right\}.$$

A convenient method of evaluating the definite integral is to plot the integrand for particular numerical values of x, y, ξ, λ. But it is to be noticed that if λ be small, say $\lambda = 0{\cdot}01$, then, except near the boundary, ψ is given by the integrated terms to a high degree of approximation.

In fig. 1 the stream lines are shown for the case $\lambda = 0{\cdot}01$, $\xi = 1$. They are drawn for the equidistant values $(\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3})\,\varpi/\pi$ of the stream function. They are, however, independent of λ.

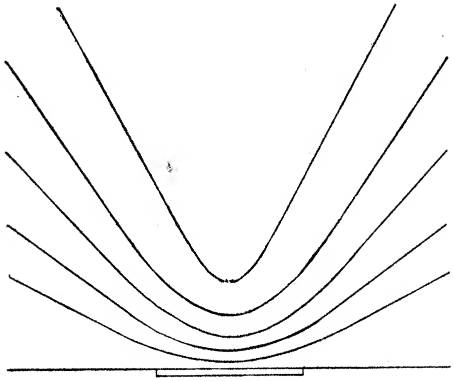

FIG. 1.

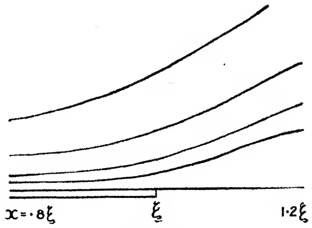

FIG. 2.

In fig. 2 the stream lines are drawn in the neighbourhood of $x = 1$, $y = 0$ for the same case. The method of evaluating the integral is similar to one to be mentioned later. The stream lines are drawn for

$$\psi = (0{\cdot}03,\ 0{\cdot}06,\ 0{\cdot}12,\ 0{\cdot}24)\,\varpi/\pi.$$

Distribution of Velocity along the Axis of x.

$$\left(\frac{\partial\psi}{\partial y}\right)_{y=0} = \frac{\varpi}{\pi}\int_0^\infty \frac{\sin k(\xi+x)+\sin k(\xi-x)}{k(1+2\lambda k)}\,dk.$$

Now $$\int_0^\infty \frac{\sin ak}{b+k}\,dk = \int_{ab}^\infty \frac{\sin(m-ab)}{m}\,dm \quad (\text{provided } a>0)$$

$$= \left(\frac{\pi}{2}-\mathrm{Si}\,(ab)\right)\cos(ab)+\mathrm{Ci}\,(ab)\sin(ab).$$

Hence

$$\left(\frac{\partial y}{\partial y}\right)_{y=0} = \varpi - \frac{\varpi}{\pi}\left[\left(\frac{\pi}{2} - \mathrm{Si}\,\theta\right)\cos\theta + \mathrm{Ci}\,\theta\sin\theta + \left(\frac{\pi}{2} - \mathrm{Si}\,\phi\right)\cos\phi + \mathrm{Ci}\,\phi\sin\phi\right], \qquad 0 < x < \xi,$$

$$= -\frac{\varpi}{\pi}\left[\left(\frac{\pi}{2} - \mathrm{Si}\,\theta\right)\cos\theta + \mathrm{Ci}\,\theta\sin\theta - \left(\frac{\pi}{2} - \mathrm{Si}\,\phi\right)\cos\phi - \mathrm{Ci}\,\phi\sin\phi\right], \qquad x > \xi,$$

where $\theta = (\xi + x)/2\lambda, \qquad \phi = |\xi - x|/2\lambda.$

The distribution of the velocity along the boundary, of which the moving section is bounded by $x = \pm 1$, is exhibited in the following table. The evaluation has been performed by the aid of Glaisher's Tables of the Sine and Cosine Integrals.

λ.	$x = 0$.	0·6.	1.	1·4.	2.	3.
0·01	0·987	0·98	0·497	0·013		
0·1	0·88	0·833	0·469	0·101	0·039	
1	0·452	0·416	0·303	0·183	0·119	0·017

The table shows the ratio of the velocity to ϖ.

In addition to the above values it is interesting to note, in the case of $\lambda = 0·01$, how rapidly the velocity changes in the neighbourhood of $x = 1$.

$x = 0·9$ .... $u = 0·937\,\varpi$,
$x = 0·99$ .... $u = 0·713\,\varpi$,
$x = 1$ .... $u = 0·497\,\varpi$.

From the distribution of the relative velocity of the sliding piece and the liquid in contact with it, the traction ($\mu\,\partial^2\psi/\partial y^2$, $y = 0$) exerted by the one on the other can be directly obtained,

$$\mu\frac{\partial^2\psi}{\partial y^2} = -\frac{\mu}{\lambda}\left(\varpi - \frac{\partial\psi}{\partial y}\right), \qquad y = 0,$$

where μ is the coefficient of viscosity.

The total traction exerted by the liquid on the sliding piece per unit of its length is given as follows:—For the values 0·01, 0·1, 1 of λ, traction/$\mu\varpi$ has the values 3·1, 1·6, 0·56, the breadth being two units.

The total rate of flow in the direction of motion of the sliding piece is $2\varpi\xi/\pi$, and this is independent of λ. This is to be explained by the fact that, although the sliding piece exerts less traction on the liquid the greater the value of λ, the liquid itself experiences less resistance from the remainder of the boundary.

Problem 2.

From the foregoing solution it is possible to derive immediately the stream lines where there are two such sliding pieces in different parts of the boundary.

In fig. 3 the stream lines are drawn for the case when the part of the boundary $1 < x < 3$ is given a velocity ϖ in the direction of x, and the part $-1 > x > -3$ is given an equal velocity in the opposite direction. The stream lines are drawn for the equidistant values $(0{\cdot}5,\ 0{\cdot}75,\ 1,\ 1{\cdot}25,\ 1{\cdot}5)\ \varpi/\pi$ of the stream function.

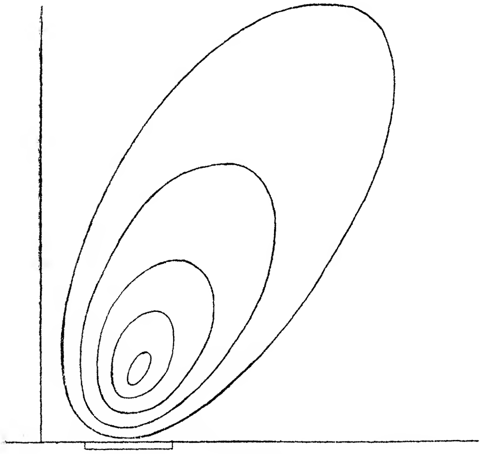

FIG. 3.

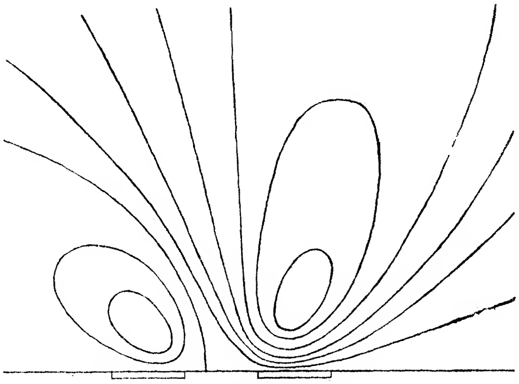

FIG. 4.

In fig. 4 the stream lines are drawn for the case in which the velocity of the second sliding piece is half that of the first. The lines are shown for the same values of ψ as in the previous case, except that $\psi = 0{\cdot}25\ \varpi/\pi$ is drawn as well.

Problem 3.

We proceed to consider the case in which the sliding piece, instead of having a uniform velocity, is given a periodic velocity $\varpi \cos pt$.

The stream function must now satisfy the equation

$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = 0,$$

where ν is the kinematic coefficient of viscosity.

The appropriate solution is

$$\psi = (\mathrm{A}e^{-ky} + \mathrm{B}e^{-my}) \cos kx e^{ipt}, \quad \text{where} \quad m^2 = k^2 + ip/\nu,$$

leading to
$$m = q + i\mu,$$

q being taken positive. Also, since we are concerned with a maintained motion, p is real.

The axis of x is a stream line, and therefore

$$A+B = 0.$$

This solution is appropriate to the harmonic distribution of velocity $\varpi_k \cos kx\, e^{ipt}$ of the boundary. Hence the remaining boundary condition is

$$\frac{\partial\psi}{\partial y}-\lambda\frac{\partial^2\psi}{\partial y^2} = \varpi_k \cos kx\, e^{ipt} \qquad (y = 0).$$

This leads to $\qquad -(k+\lambda k^2)\,A-(m+\lambda m^2)\,B = \varpi_k.$

Hence $\qquad A = -B = \varpi_k/(m-k+i\lambda p/\nu).$

Taking real parts, we find

$$\psi = \varpi_k F(k)\cos kx,$$

where

$$F(k) = [e^{-ky}(P\sin pt+Q\cos pt)-e^{-qy}\{P(\cos\mu y\sin pt-\sin\mu y\cos pt)$$
$$+Q(\cos\mu y\cos pt+\sin\mu y\sin pt)\}]/(P^2+Q^2),$$

where $\qquad P = \mu+\lambda p/\nu, \qquad Q = q-k.$

The solution, following the notation of Problem 1, is

$$\psi = \frac{\varpi}{\pi}\int_0^\infty [\sin k(\xi+x)+\sin k(\xi-x)]\frac{F(k)}{k}\,dk.$$

The work of calculating the stream lines for particular values of the constants involved is a matter of some complexity. I have performed it for one case, but not sufficiently to give more than a rough idea of their form. But it was possible to see that a vortex is periodically formed and destroyed in the neighbourhood of the sliding piece in this particular case. It would be interesting to trace the stream lines for a range of values of the time, and so exhibit the growth and decay of this vortex. But at the present time I have not the requisite leisure.

Distribution of Velocity over the Sliding Piece.—It can easily be shown that

$$\left(\frac{\partial\psi}{\partial y}\right)_{y=0} = \frac{\varpi}{\pi}\int_0^\infty [\sin k(\xi+x)+\sin k(\xi-x)]\times\left[\left(1-\frac{P\lambda p/\nu}{P^2+Q^2}\right)\cos pt\right.$$
$$\left.+\frac{Q\lambda p/\nu}{P^2+Q^2}\sin pt\right]\frac{dk}{k}.$$

The numerical evaluation of this integral presents features of some interest. Apart from x the variables are ξ, λ, p/ν; ξ has been taken equal to one unit as before, $\lambda = 0{\cdot}01$, and the integral has been evaluated for

$$p/\nu = 10^{-1},\ 1,\ 10,\ 10^2,\ 10^3,\ 10^4,\ 10^6.$$

The method of evaluation depends upon the magnitude of p/ν, and one or two illustrations of the methods employed will be given.

(*a*). $p/\nu = 0{\cdot}1$.

$$\int_0^\infty \frac{\mathrm{P}}{\mathrm{P}^2+\mathrm{Q}^2} \sin k(\xi+x)\frac{dk}{k} = \int_0^2 \frac{\mathrm{P}}{\mathrm{P}^2+\mathrm{Q}^2} \sin k(\xi+x)\frac{dk}{k} + \int_2^\infty \frac{\sin k(\xi+x)}{\mathrm{P}}\frac{dk}{k},$$

since Q is very small for $k > 2$. At the same time $\mu = p/2k\nu$, so that the second integral is capable of evaluation by the aid of the tables of sine and cosine integrals as in similar integrals in Problem 1. The first integral can be evaluated by plotting the integrand.

In this same case the coefficient of sin pt can be evaluated by plotting the integrand, since it decreases rapidly with k.

(*b*). $p/\nu = 10$.

The coefficient of $\varpi \cos pt$ (for $x = 0$) involves an integral which can be replaced by

$$\int_0^6 (0{\cdot}223 + 0{\cdot}105\,k + 0{\cdot}0063\,k^2)\frac{\sin k}{10k}\,dk + \int_6^\infty \frac{\sin k\,dk}{50+k}$$

to a good degree of accuracy. These integrals are easily evaluated.

(*c*). $p/\nu = 10^4$.

The coefficient of $\varpi \cos pt$ (for $x = 0$) involves an integral which can be replaced by

$$\int_0^{100} (0{\cdot}005 + 0{\cdot}0000184\,k)\frac{\sin k}{k}\,dk.$$

Or (*d*). $p/\nu \geq 10^4$.

$$\left(\frac{\partial\psi}{\partial y}\right)_{y=0} = \varpi\left[\left(1-\frac{\mathrm{P}\lambda p/\nu}{\mathrm{P}^2+\mathrm{Q}^2}\right)\cos pt + \frac{\mathrm{Q}\lambda p/\nu}{\mathrm{P}^2+\mathrm{Q}^2}\sin pt\right],$$

for any value of x between $\pm\xi$, except near these limits.

When p/ν is very great $\left(\frac{\partial\psi}{\partial y}\right)_{y=0} = 0$.

Representing the velocity at the centre of the sliding piece by $\mathrm{X}\cos pt + \mathrm{Y}\sin pt$, we have the following values for X and Y.

p/ν ……	10^{-1}.	1.	10.	10^2.	10^3.	10^4.	10^6.
X ………	0·99	0·99	0·97	0·93	0·79	0·5	0·07
Y ………	0·001	0·005	0·025	0·06	0·15	0·21	0·06

X decreases from 1 to 0, Y increases from 0 to a maximum value and then decreases again to zero. The velocity at the point $x = \frac{1}{2}\xi$ is 1 or 2 per cent.

below that at the centre, and at $x = \xi$, X and Y have practically half their values at the centre.

The maximum (as regards the time) integral traction on the sliding piece per unit of its length can easily be found. It has the following values in units of the system of measurements employed.

p/ν	10^{-1}.	1.	10.	10^2.	10^3.	10^4.	10^6.
$\dfrac{\text{Traction}}{\mu\varpi}$	3	3	14	23	55	114	190

For water, $p/\nu = 10^3$ corresponds to a period of 0·6 sec. in C.G.S. units, so that for this period the traction is about 18 times that for steady motion.

Problem 4.

This is the same as the first problem, except that the liquid is further bounded by $y = b$, which is fixed. The corresponding boundary condition is taken to be that no relative motion is possible over it.

A possible stream function is given by

$$\psi = \{(\mathrm{A}+\mathrm{C}y)\sinh ky + (\mathrm{B}+\mathrm{D}y)\cosh ky\}\cos kx.$$

The conditions to be satisfied are

$$\psi = 0, \qquad \frac{\partial\psi}{\partial y} - \lambda\frac{\partial^2\psi}{\partial y^2} = \varpi_k \cos kx \qquad (y = 0);$$

$$\psi = 0, \qquad \frac{\partial\psi}{\partial y} = 0 \quad (y = b).$$

These lead to $\mathrm{B} = 0$, and

$$\frac{\mathrm{A}}{-kb^2} = \frac{\mathrm{C}}{kb - \frac{1}{2}\sinh 2kb} = \frac{\mathrm{D}}{\sinh^2 kb}$$
$$= \frac{\varpi_k}{-k^2b^2 - 2k^2b\lambda + k\lambda\sinh 2kb + \sinh^2 kb} = \frac{\varpi_k}{\phi(k)}.$$

Using Fourier's theorem, we obtain the stream function in the form

$$\psi = \frac{\varpi}{\pi}\int_0^\infty [\sin k(\xi+x) + \sin k(\xi-x)]\frac{\chi(k)}{\phi(k)}\frac{dk}{k},$$

where

$$\chi(k) = [-kb^2 + (kb - \tfrac{1}{2}\sinh 2kb)\,y]\sinh ky + y\sinh^2 kb\cosh ky.$$

In fig. 5 the stream lines are drawn for the case of $\xi = 1$, $b = 2$, $\lambda = 1$. Since calculating ψ for the unpractical case of $\lambda = 1$, I have lighted upon a method of evaluating the integral for $\lambda = 0{\cdot}01$, but the advantage of recalculation is not great in the case of b so great as 2.

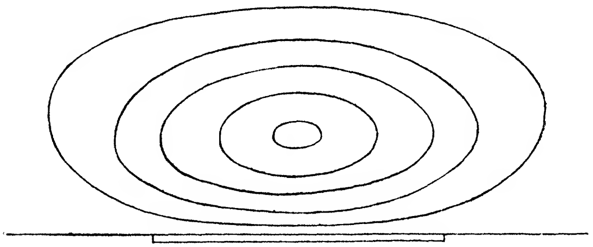

FIG. 5.

Velocity of the Liquid at the Centre of the Sliding Piece.—The expression for the velocity of the liquid along the boundary $y = 0$ is

$$\left(\frac{\partial\psi}{\partial y}\right)_{y=0} = \frac{\varpi}{\pi}\int_0^\infty \frac{-k^2b^2+\sinh^2 kb}{k\phi(k)}\left[\sin k(\xi+x)+\sin k(\xi-x)\right] dk.$$

The ratio of the velocity at the centre of the sliding piece to ϖ is given below :—

λ.	$b = 1$.	0·1.	0·01.
0·01	0·96	0·72	0·2
0·1	0·71	0·2	0·025

When b is small, it is easily shown that the velocity at any point of the sliding piece is $b\varpi/(b+4\lambda)$, except that the approximation breaks down when $\xi-x$ is small, and that at $x = \xi$ the velocity drops to half that at the centre.

Hence the traction exerted by the liquid on the sliding piece is (per unit length)

$$\frac{2\varpi\xi}{\lambda}\left(1-\frac{b}{b+4\lambda}\right)\mu = \frac{8\mu\varpi\xi}{b+4\lambda}.$$

As b is decreased indefinitely, this approaches the limit $2\mu\varpi\xi/\lambda$. This is the same limit as in the case of the rapid periodic motion of the last case.

Problem 5.

The liquid is bounded by two parallel planes as in the previous case, but the sliding piece is given a periodic velocity $\varpi \cos pt$.

The appropriate typical harmonic solution is

$$\psi = (\mathrm{A}\cosh ky+\mathrm{B}\sinh ky+\mathrm{C}\cosh my+\mathrm{D}\sinh my)\cosh kx\, e^{ipt},$$

with the same notation as in Problem 3.

The solution follows as in previous cases, and I proceed at once to discuss the velocity of the liquid at points on the sliding piece.

Consider $\int_0^\infty \frac{\sin k}{k} f(k)\, dk.$

If $f(k)$ vary slowly with k within the range $0 < k < 20$ (say), the value of the integral is $\frac{1}{2}\pi f(0)$ to a very fair degree of accuracy.

It is assumed that $f(k)$ is decreasing and one-signed. The rate of variation need not be very slow for the approximation to hold, for example $f(k)$ may decrease practically to zero in the interval. The approximations which follow are based on this consideration, but it is not claimed that the intermediate results in the range are more than roughly correct.

$$\left(\frac{\partial\psi}{\partial y}\right)_{y=0} = \frac{\varpi}{\pi}\int_0^\infty [\sin k(\xi+x)+\sin k(\xi-x)]\frac{f(k)}{k}\, dk,$$

where $f(k)$ is the real part of e^{ipt} divided by

$$1+\frac{i\lambda p}{\nu}\cdot\frac{m\sinh kb\cosh mb-k\cosh kb\sinh mb}{2km(1-\cosh kb\cosh mb)+(k^2+m^2)\sinh kb\sinh mb}.$$

Now, provided b be small, the variation in $f(k)$ is small for moderate values of k. Accordingly $f(k)$ is replaced by $f(0)$.

Now, when $k = 0$, $m = q+i\mu$, where $q = \mu = (p/2\nu)^{\frac{1}{2}} = \alpha$ (say).

Put $\alpha b = \eta$, $\lambda\alpha = \zeta$.

It is found that

$$\left(\frac{\partial\psi}{\partial y}\right)_{y=0,\ x=0} = \varpi(\mathrm{R}\cos pt+\mathrm{S}\sin pt)/\mathrm{T},$$

where

$$\begin{aligned}\mathrm{R} = {} & 2-4\cosh\eta\cos\eta-2\eta\zeta\sinh\eta\sin\eta+\cosh 2\eta+\cos 2\eta \\ & -(2\eta-\zeta)(\cosh\eta\sin\eta-\sinh\eta\cos\eta)+\tfrac{1}{2}(2\eta-\zeta+\eta^2\zeta)\sin 2\eta \\ & -\tfrac{1}{2}(2\eta-\zeta-\eta^2\zeta)\sinh 2\eta+\tfrac{1}{2}\eta(\eta-\zeta)(\cosh 2\eta-\cos 2\eta),\end{aligned}$$

$$\begin{aligned}\mathrm{S} = {} & \zeta[\eta(\cosh 2\eta+\cos 2\eta-2\cosh\eta\cos\eta)-\tfrac{1}{2}(1+\eta^2)\sinh 2\eta \\ & -\tfrac{1}{2}(1-\eta^2)\sin 2\eta+(\cosh\eta\sin\eta+\sinh\eta\cos\eta)],\end{aligned}$$

$$\begin{aligned}\mathrm{T} = {} & 2-4\cosh\eta\cos\eta-4\eta\zeta\sinh\eta\sin\eta-2(\eta-\zeta)(\cosh\eta\sin\eta \\ & -\sinh\eta\cos\eta)+(\eta-\zeta)(1+\eta\zeta)\sin 2\eta-(\eta-\zeta)(1-\eta\zeta)\sinh 2\eta \\ & +(1+\eta^2\zeta^2)(\cosh 2\eta+\cos 2\eta)+\tfrac{1}{2}(\eta-\zeta)^2(\cosh 2\eta-\cos 2\eta).\end{aligned}$$

(*a*) Approximation when αb is small.—

The velocity at the centre of the sliding piece is equal to $\frac{b}{b+4\lambda}\varpi\cos pt$, the coefficient being the same as in the case of steady motion.

This approximation is valid if (1) $b = 0{\cdot}1$ cm., $\alpha \leq 1$, which gives for water a period of 5 mins. and over, (2) $b = 0{\cdot}01$ cm., $\alpha \leq 10$, which gives for water a period of 3 sec. and over.

(b) Approximation when $\alpha b \geqq 4$, so that $\cosh \alpha b$ and $\sinh \alpha b$ can be neglected, compared with $\cosh^2 \alpha b$, etc.—

This applies in the case of water to periods of 0·2 second and under, and 0·002 second and under, for the cases $b = 0·1$ and $b = 0·01$ respectively.

The velocity at the centre of the sliding piece is

$$\varpi \left[\left\{1-\tfrac{1}{2}(2\eta-\zeta-\eta^2\zeta)+\tfrac{1}{2}\eta(\eta-\zeta)\right\} \cos pt \right.$$
$$\left. +\left\{\eta\zeta-\tfrac{1}{2}\zeta(1+\eta^2)\right\} \sin pt\right] / \left[(1+\eta^2\zeta^2)+\tfrac{1}{2}(\eta-\zeta)^2-(\eta-\zeta)(1-\eta\zeta)\right].$$

The following table includes the whole range of values of αb, for $\lambda = 0·01$ $b = 0·01$, in which case $\eta = \zeta$, and the above formula is simplified.

αb......	0·01.	0·1.	1.	2.	3.	4.	5.	6.	10.
R/T......	0·2	0·2	0·199	0·19	0·157	0·121	0·097	0·083	0·05
−S/T......	0	0	0·011	0·088	0·082	0·07	0·064	0·056	0·05

It follows that the traction exerted on the sliding piece bears a ratio to that exerted in the steady motion which lies between 1 and $(b + 4\lambda)/4\lambda$.

It will be noticed in Problem 3 when the period of the motion is very small, and in both of the last two problems when b is small, that the condition of no slip at the boundary is not approximately satisfied for $\lambda = 0·01$. If λ be taken sufficiently small, then it is easily shown that the maximum total traction per unit length on the moving piece has a value $2\mu\varpi\xi(p/\nu)^{\frac{1}{2}}$, in the case of very rapid motion in Problem 3. In the steady motion of Problem 4, when the distance b between the two planes is very small, the integral traction has the value $8\mu\varpi\xi/b$. In the periodic motion of Problem 5, if b be small and $b(p/2\nu)^{\frac{1}{2}} < 0·1$, the integral traction depends upon b and not upon p, and has the maximum value $8\mu\varpi\zeta/b$. If, however, b be small and p sufficiently great, the traction is independent of b and has the value $2\mu\varpi\xi(p/\nu)^{\frac{1}{2}}$. This can be shown from the approximate expressions for R/T, S/T, which are given above.

Thus in the case of periodic motion between two near parallel planes the integral traction is equal to or greater than $8\mu\varpi\xi/b$, and less than or equal to $2\mu\varpi\xi(p/\nu)^{\frac{1}{2}}$, provided this second limit is greater than the first.